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Pointwise Approximation by Algebraic Polynomials

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In 1951, A.F. Timan [9] proved that the approximation of smooth functions by algebraic polynomials can by improved near the end points of the approximation interval. He proved that for all $f \in C^r[-1, 1]$, $r \in \mathbb{N}_0$ there exists a polynomial of degree not greater than n, $p_n \in \Pi_n$ such that for all $x \in [-1, 1]$

$$|f(x) - p_n(x)| \leqslant C\, \Delta_n(x)^r \omega_1(f^{(r)}, \Delta_n(x)), \tag{1}$$

where $\Delta_n(x) = \sqrt{1-x^2}/n + 1/n^2$, and $\omega_1(f^{(r)}, \cdot)$ is the modulus of continuity of the rth derivative of f. Subsequently, Brudnyi [1] extended the result by replacing $\omega_1(f^{(r)}, \Delta_n(x))$ by the sth modulus of continuity $\omega_s(f^{(r)}, \Delta_n(x))$ $(s \in \mathbb{N})$ while Gopengauz [5] proved the estimation simultaneously for all derivatives of f; i.e., we have

$$|f^{(k)}(x) - p_n^{(k)}(x)| \leqslant C\, \Delta_n(x)^{r-k} \omega_s(f^{(r)}, \Delta_n(x)) \tag{2}$$

for $0 \leqslant k \leqslant r$.

In 1963, at the Oberwolfach conference on approximation theory, G. G. Lorentz [11] raised the question of whether $\Delta_n(x)$ in (1) can be replaced by $\Gamma_n(x) = \sqrt{1-x^2}/n$. This question was answered positively in 1966 by Telyakowskii [8]. Subsequently, attempts were made to obtain the same generalization of this result as in (2). Gopengauz [4] obtained (2) with $\Gamma_n(x)$ instead of $\Delta_n(x)$ for $s=1$, De Vore [2] proved the result for $s=2$ and $r=0$, and Hinnemann and Gonska [6] proved the case $s=2$, $r \geqslant 0$ and $k=0$.

In 1985, considerable progress was made in papers by Gonska and Hinnemann [3] and Yu [10]. Gonska and Hinnemann proved by two independent proofs the cases $s \leqslant r+2$, $k=0$ and $s \leqslant r$, $0 \leqslant k \leqslant r-s$, while Yu was the first to show by a counterexample that (2), with $\Delta_n(x)$ replaced by $\Gamma_n(x)$, in general does not hold. He gave a counterexample for the case

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$s=r+3$ (and therefore also for $s \geqslant r+3$). Combining both papers, for $k=0$ the conjecture therefore holds if and only if $s \leqslant r+2$.

By extending the proof methods of Gonska and Hinnemann and Yu we prove in this paper that (2) holds, with $\Delta_n(x)$ replaced by $\Gamma_n(x)$, if and only if $0 \leqslant k \leqslant \min\{r-s+2, r\}$, which solves the problem completely.

THEOREM 1. *Let $r, s \in \mathbb{N}_0$. There exists a constant $C_{r,s} \in \mathbb{R}$ such that for all $f \in C^r[-1, 1]$ and all $n \geqslant \max\{4(r+1), r+s\}$ there exists a $p_n \in \Pi_n$ with*

$$|f^{(k)}(x) - p_n^{(k)}(x)| \leqslant C_{r,s} \Gamma_n(x)^{r-k} \omega_s(f^{(r)}, \Gamma_n(x)) \qquad (3)$$

for all $k \in \mathbb{N}_0$ with $0 \leqslant k \leqslant \min\{r-s+2, r\}$ and all $x \in [-1, 1]$.

Proof. Since the case $s=1$ implies the case $s=0$ we assume $s \geqslant 1$. Theorems 4.2 and 5.4 of Gonska and Hinnemann [3] imply the existence of linear polynomial operators $Q_n = Q_n^{(r,s)}: C^r[-1, 1] \to \Pi_n$ with

$$(Q_n f)^{(k)}(\pm 1) = f^{(k)}(\pm 1) \qquad \text{for all} \quad f \in C^r[-1, 1] \text{ and } 0 \leqslant k \leqslant r, \qquad (4)$$

$$|f^{(k)}(x) - (Q_n f)^{(k)}(x)| \leqslant A_{r,s} \Delta_n(x)^{r+s-k} \|f^{(r+s)}\|$$
$$\text{for all} \quad f \in C^{r+s}[-1, 1], |x| \leqslant 1, \text{and } 0 \leqslant k \leqslant r+s, \qquad (5)$$

and

$$|f^{(k)}(x) - (Q_n f)^{(k)}(x)| \leqslant A_{r,s} \Delta_n(x)^{r-k} \omega_s(f^{(r)}, \Delta_n(x))$$
$$\text{for all} \quad f \in C^{(r)}[-1, 1], |x| \leqslant 1, \text{and } 0 \leqslant k \leqslant r. \qquad (6)$$

By using these relations we now prove the assertion. In case $\sqrt{1-x^2} \geqslant n^{-1}$, $\Delta_n(x) \leqslant 2\Gamma_n(x)$ and (6) imply (3). Suppose now that x is fixed with $0 < x < 1$ and $\sqrt{1-x^2} < n^{-1}$ $(-1 < x < 0$ is treated analogously). The result of Müller [7] implies that for $f \in C^r[-1, 1]$ there exists a $F_x \in C^{r+s}[-1, 1]$ with

$$\|f^{(k)} - F_x^{(k)}\| \leqslant c_{r,s} \Gamma_n(x)^{r-k} \omega_s(f^{(r)}, \Gamma_n(x)) \qquad (7)$$

and

$$\Gamma_n(x)^s \|F_x^{(r+s)}\| \leqslant c_{r,s} \omega_s(f^{(r)}, \Gamma_n(x)). \qquad (8)$$

Thus we obtain as an upper bound of $|f^{(k)}(x) - (Q_n f)^{(k)}(x)|$

$$|f^{(k)}(x) - F_x^{(k)}(x)| + |F_x^{(k)}(x) - (Q_n F_x)^{(k)}(x)| + |\{Q_n(F_x - f)\}^{(k)}(x)|. \qquad (9)$$

By (7), the first term has the required upper bound. The second term is equal to (note that the following constants are not the same in each step)

$$\left| \int_x^1 \int_{u_1}^1 \cdots \int_{u_{r-k}}^1 (F_x - Q_n F_x)^{(r+1)}(u_{r-k+1})\, du_{r-k+1} \cdots du_1 \right|$$
$$\leqslant A_{r,s}(1-x)^{r-k+1} \Delta_n(x)^{s-1} \|F_x^{(r+s)}\| \qquad \text{by using (5).}$$

Since $\Delta_n(x) \leqslant 2n^{-2}$ this is less than

$$C_{r,s}\Gamma_n(x)^{r-k}\omega_s(f^{(r)}, \Gamma_n(x))\,(n\sqrt{1-x^2})^{r-s-k+2}.$$

If $k \leqslant r-s+2$, the last factor is bounded by one. To estimate the last term in (9) we consider

$$\int_x^1\int_{u_1}^1\cdots\int_{u_{r-k-1}}^1 \{Q_n(F_x-f)\}^{(r)}(u_{r-k})\,du_{r-k}\cdots du_1 \tag{10}$$

$$= \sum_{\nu=0}^{r-k-1} (-1)^{r-k-1-\nu}\frac{(1-x)^\nu}{\nu!}$$
$$\times \{Q_n(F_x-f)\}^{(k+\nu)}(1) + (-1)^{r-k}\{Q_n(F_x-f)\}^{(k)}(x). \tag{11}$$

Equation (7) and $1-x \leqslant \Gamma_n(x)$ imply that the first term of (11) is bounded by $C_{r,s}\Gamma_n(x)^{r-k}\omega_s(f^{(r)}, \Gamma_n(x))$. By using (6) we get as an upper bound of (10)

$$C_{r,s}(1-x)^{r-k}[\omega_s(F_x^{(r)}-f^{(r)}, \Delta_n(x)) + \|F_x^{(r)}-f^{(r)}\|]$$
$$\leqslant C_{r,s}\Gamma_n(x)^{r-k}\,\|F_x^{(r)}-f^{(r)}\|$$

which with (7) gives the result.

THEOREM 2. *Let $r, s \in \mathbb{N}_0$. For all $C \in \mathbb{R}$ and all $n \in \mathbb{N}$ there exists a $f \in C^r[-1, 1]$ such that for all $p_n \in \prod_n$ there exists a $x = x_k \in [-1, 1]$ with*

$$|f^{(k)}(x) - p_n^{(k)}(x)| > C\Gamma_n(x)^{r-k}\omega_s(f^{(r)}, \Gamma_n(x))$$

for all $k \in \mathbb{N}_0$ with $r-s+3 \leqslant k \leqslant r$.

Proof. We only give a sketch. We first assume $s = r+3$ and define as in Yu [10]

$$f_{r,a}(x) = \begin{cases} (-1+a-x)^{2r+3}, & -1 \leqslant x \leqslant -1+a \\ 0, & -1+a < x \leqslant 1 \end{cases}$$

with $a = \{4Cn^{2r+2}\}^{-1}$. The assertion now follows by arguments similar to those given in the proof of Theorems 1 and 2 of Yu [10]. If $s > r+3$ we obtain the result from above since $\omega_s(f^{(r)}, \cdot) \leqslant 2^{s-r-3}\omega_{r+3}(f^{(r)}, \cdot)$. If $s < r+3$ we consider the function

$$f(x) = \int_{-1}^x \int_{-1}^{u_{r-s+3}} \cdots \int_{-1}^{u_2} f_{s-3,a}(u_1)\,du_1 \cdots du_{r-s+3}$$

with $a = \{4Cn^{2s-4}\}^{-1}$ and obtain again from the special case $s = r+3$ the assertion.

Summarizing the results of both theorems we have the following situation. Given r, s we can find by Theorem 1 a constant $C_{r,s}$ such that (3) holds simultaneously for the first $r-s+2$ derivatives. For the same constant we can then find by Theorem 2 a function f such that (3) is wrong for all higher derivatives. Especially, we have the following corollary.

COROLLARY. *Assertion* (3) *holds simultaneously for all r derivatives if and only if $s \leqslant 2$.*

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